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Michelle S. Berry[1], Victoria Diaz[2], Brandy Doleshal[3], Taylor Martin[3], Emily T. Winn[4], and Max Zhou[5]

[1]Goucher College
[2]Florida International University
[3]Sam Houston State University
[4]College of the Holy Cross
[5]Indiana University Bloomington

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Abstract. A link is a smooth embedding of a finite number of disjoint copies of S^1 into S^3. Links of one component are known as knots. We are particularly interested in twisted torus links, as no complete classification of them currently exists. Specifically, we are interested in finding ways to determine the number of components in a twisted torus link. Twisted torus links of one component are especially interesting because there exist more ways to classify knots than there are ways to classify links. In this paper, we will examine patterns in the parameters of a twisted torus link that reveal general and specific information about components.

1. Introduction

A *knot* is a smooth embedding of a circle, S^1 into S^3, and a *link* is a finite collection of disjoint knots. If a knot K is part of a link L then we say that K is a component of L. A knot can be thought of as a link of one component. The study of knots and links is a rich area of low dimensional topology with applications to other areas of mathematics. One simple family of links is the collection of *torus links*, which are curves that can be embedded on a torus. The torus link $T(p,q)$ intersects the meridian of the torus p times and the longitude q times. Because there is an automorphism of S^3 that switches the meridian and the longitude of the torus, the torus links $T(p,q)$ and $T(q,p)$ are *isotopic*, meaning one can be deformed into the other through such an automorphism. When p and q are co-prime, $T(p,q)$ is a knot. A much more interesting, not well-understood family of links is that of *double torus links*, which are links that can be embedded on an orientable genus two surface. In this paper, we study a generalization of double torus links called *twisted torus links*, which we will define below.

* *Corresponding author*

One avenue for studying these types of links is by considering their braid representations. A *braid* on n strands is the union of n non-intersecting paths in the cylinder $S^1 \times [0,1]$ that all start at one of n points in the circle $S^1 \times 0$ and end in one of the same n points in the circle $S^1 \times 1$ such that each cross-section $S^1 \times i$ contains exactly n points. This means that all the paths are always descending as they move through the cylinder. The closure of a braid is defined by joining the i-th point in $S^1 \times 0$ to the i-th point in $S^1 \times 1$. Figure 1 depicts a braid and its closure. Every link can be expressed as the closure of a braid [1, p. 129]. The interested reader can learn more about knots, links, and braids in the book *The Knot Book* by Colin Adams [1].

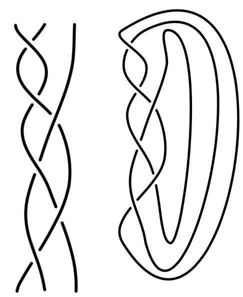

FIGURE 1. A 3-strand braid and its closure.

A braid on n strands can be described as a series of crossings on two consecutive strands. σ_i denotes the i^{th} strand passing over the $i-1^{st}$ strand, and σ_i^{-1} denotes the i^{th} strand passing under the $i-1^{st}$ strand, which, if occurring immediately before or after σ_i, is equivalent to the i^{th} and $i-1^{st}$ strands having no crossings. Given two braids on n strands β_1 and β_2, we can form a new braid β_3 by concatenating the braids, or placing the top of β_2 under the bottom of β_1 so that β_3 is the braid consisting of first β_1 followed by β_2. We use $\beta_3 = \beta_1 \cdot \beta_2$ as shorthand for concatenation. Braids on n strands form a group under concatenation, which we denote B_n. The identity element is the braid with no crossings, its generators are the σ_i, and the inverse of σ_i is σ_i^{-1}. A group presentation for B_n is

$$B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ when } |i-j| \geq 2 \rangle.$$

A specific sequence of crossings can be grouped together and called a twist. A single positive twist on n strands is denoted by the braid word $\sigma_1 \cdots \sigma_{n-1}$ and a single negative twist is given by the braid word $\sigma_1^{-1} \cdots \sigma_{n-1}^{-1}$. A full positive twist on n strands is n single positive twists next to each other or $(\sigma_1 \cdots \sigma_{n-1})^n$ and a full negative twist on n strands is n single negative twists next to each other or $(\sigma_1^{-1} \cdots \sigma_{n-1}^{-1})^n$. We note that a full positive twist and a full negative twist are inverses of each other, though a single positive twist and a single negative twist are not inverses.

The $T(p,q)$ torus link can be represented as a q-strand braid with p positive twists. We can also let p be negative in which case it is a braid on q strands with $|p|$ negative twists. This corresponds to having the knot wrap around the torus in the opposite direction. The braid word for the $T(p,q)$ torus link is $(\sigma_1 \ldots \sigma_{q-1})^p$.

We define the *twisted torus link*, $T((r_1, s_1), (r_2, s_2))$, using the definition given by Birman and Kofman [2], to be the closure of the r_2-strand braid

$$T((r_1, s_1), (r_2, s_2)) = (\sigma_1 \sigma_2 \ldots \sigma_{r_1 - 1})^{s_1} (\sigma_1 \sigma_2 \ldots \sigma_{r_2 - 1})^{s_2}$$

where $2 \leq r_1 \leq r_2$. Note that Birman and Kofman impose the condition that s_1 and s_2 are positive; we allow these parameters to be negative. We can describe $T((r_1, s_1), (r_2, s_2))$ as the closure of the braid consisting of s_1 twists on r_1 strands followed by s_2 twists on r_2 strands. Figure 2 shows the twisted torus link $T((3,2),(5,4))$ and the general twisted torus link $T((r_1, s_1), (r_2, s_2))$ is depicted in Figure 3.

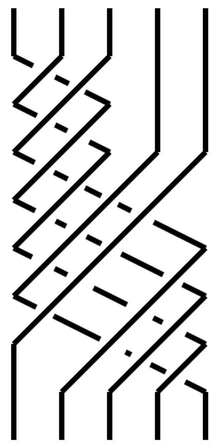

FIGURE 2. The twisted torus link $T(3,2),(5,4))$

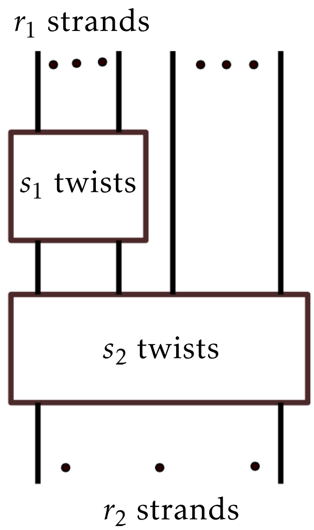

FIGURE 3. The twisted torus link $T((r_1, s_1), (r_2, s_2))$

Much of the recent literature in this area focuses on twisted torus knots. For example Lee [5] gives a characterization of when a twisted torus knot is unknotted. Morimoto shows that there are infinitely many composite twisted torus knots [6]. Dean classifies primitive/middle-Seifert-fibered twisted torus knots in [3]. However, in general, it is difficult to determine when a twisted torus link is actually a knot. Each of the above sources restricts their attention to twisted torus links $T((r_1,s_1),(r_2,s_2))$ where r_2 and s_2 are coprime and s_1 is a multiple of r_1; these are sufficient but not necessary conditions to guarantee that $T((r_1,s_1),(r_2,s_2))$ is a knot. This begs the question, "Under what conditions is a twisted torus link $T((r_1,s_1),(r_2,s_2))$ a knot?"

The goal of this paper is to characterize relationships between the parameters of a twisted torus link $T((r_1,s_1),(r_2,s_2))$ and its number of components. In Section 2, we give necessary and sufficient conditions on r_1, r_2, s_1, and s_2 so that $T((r_1,s_1),(r_2,s_2))$ is a link with an even number of components (and therefore not a knot). In Section 3, we show that under certain conditions on r_1 and r_2 and when $s_1 = \pm s_2$, we can explicitly determine the number of components of $T((r_1,s_1),(r_2,s_2))$. Finally, in Section 4, we discuss conjectures about the component number of certain families of twisted torus links for which we have computationally verified tens of thousands of examples.

2. The Parity of the Number of Components

The braid representation of every twisted torus link $T((r_1,s_1),(r_2,s_2))$ can be associated with a permutation in S_{r_2}, the symmetric group on r_2 elements. To each of the braid group generators $\{\sigma_i\} \subseteq B_{r_2}$, we associate a corresponding transposition in S_{r_2}. For all $i \in \{1,\dots,r_2-1\}$, we associate both σ_i and σ_i^{-1} to the transposition $(i,i+1)$. Then, each twisted torus link $T((r_1,s_1),(r_2,s_2))$ is associated to the product of transpositions as determined by its braid word $(\sigma_1\sigma_2\dots\sigma_{r_1-1})^{s_1}(\sigma_1\sigma_2\dots\sigma_{r_2-1})^{s_2} \in B_{r_2}$.

This product of transpositions can be written as a product of disjoint cycles in S_{r_2}, where the cycles describe the permutation of strands in the braid. Thus, the number of disjoint cycles, where we include cycles of length one in the count, corresponds to the number of components of the twisted torus link. In this section, we determine the parity of the component number of the twisted torus link $T((r_1,s_1),(r_2,s_2))$.

Theorem 2.1. *A twisted torus link $T((r_1,s_1),(r_2,s_2))$ has an even number of components if and only if any of the following hold:*

(1) $r_1 \equiv 1 \pmod 2, r_2 \equiv 0 \pmod 2, s_2 \equiv 0 \pmod 2$

(2) $r_1 \equiv 0 \pmod 2, r_2 \equiv 0 \pmod 2, s_1 \equiv s_2 \pmod 2$

(3) $r_1 \equiv 0 \pmod 2, r_2 \equiv 1 \pmod 2, s_1 \equiv 1 \pmod 2$

Before starting the proof of Theorem 2.1 we prove a few necessary results about the symmetric group. First, recall that any permutation $\rho \in S_n$ can be written non-uniquely as a product of transpositions, and the parity of the number of transpositions is invariant. Thus, a permutation is called *even* if it requires an even number of transpositions and *odd* otherwise. Also note that any permutation in S_n can be written as a product of disjoint cycles, where the representation is unique up to the order of the disjoint cycles, which commute in S_n, and the choice of the first element listed in a cycle. We define the *component number* of a permutation to be the number of disjoint cycles in the permutation, where we include cycles of length one in this count. We denote the component number of $\rho \in S_n$ by $\#\rho$.

It is a standard exercise in abstract algebra to show that composing a permutation with a transposition changes the parity of the permutation. We prove a similar result that shows that composing a permutation with a transposition changes the parity of the component number.

Lemma 2.2. *Let $\rho, \tau \in S_n$, where τ is a transposition. Then, $\#(\rho \circ \tau) \equiv \#\rho + 1 \pmod 2$.*

Proof. We denote the transposition $\tau \in S_n$ as $\tau = (a, b)$. We consider the permutation $\rho \in S_n$ as a product of disjoint cycles, including cycles of length one, and we note that we have two cases to consider: either a and b belong to the same cycle or different cycles of ρ.

In the first case, we consider when a, b belong to the same cycle of ρ. We may write this cycle as (i_1, a, i_2, b) where i_1, i_2 stand for arbitrary disjoint blocks of integers in the cycle. Note that either or both of i_1, i_2 may be empty. Then, observe that the composition $(i_1, a, i_2, b) \circ (a, b) = (a, i_1)(b, i_2)$, so $\#(\rho \circ \tau) = \#\rho + 1$. This composition breaks the cycle containing a and b into two disjoint cycles.

In the second case, we assume that a and b belong to distinct cycles of ρ. We write these cycles as (i_1, a) and (i_2, b) where i_1, i_2 stand for arbitrary disjoint blocks of integers that may also be empty. Then, observe that the composition $[(i_1, a)(i_2, b)] \circ (a, b) = (a, i_2, b, i_1)$, so $\#(\rho \circ \tau) = \#\rho - 1$. □

We can generalize this result to determine the parity of the component number of any two permutations. This will be of use in counting components of twisted torus links.

Lemma 2.3. *Let σ, ρ be permutations in S_n. Then,*

$$\#(\rho \circ \sigma) \equiv \begin{cases} \#\rho + 1 \pmod 2 & \text{if } \sigma \text{ is odd} \\ \#\rho \pmod 2 & \text{if } \sigma \text{ is even} \end{cases}$$

Proof. In the case that σ is an odd permutation, we know that we can write σ as a product of an odd number of transpositions, $\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$, where n is odd. By associativity in S_n, we see that $\rho \circ \sigma = \rho \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$, and from Lemma 2.2, we know that each transposition changes the parity of $\#\rho$. Since there are an odd number of transpositions, the final effect is that $\#(\rho \circ \sigma) \equiv \#\rho + 1 \pmod 2$.

Similarly, if σ is an even permutation, we know that we can write σ as a product of an even number of transpositions, $\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$, where m is even. By associativity in S_n, we see that $\rho \circ \sigma = \rho \circ \tau_1 \circ \tau_2 \circ \cdots \circ \tau_m$, and from Lemma 2.2, we know that each transposition changes the parity of $\#\rho$. Since there are an even number of transpositions, the final effect is that $\#(\rho \circ \sigma) \equiv \#\rho \pmod 2$. □

Therefore, the parity of $\#(\rho \circ \sigma)$ is determined by the parities of σ and $\#\rho$. In particular, $\#(\rho \circ \sigma)$ is even if and only if one of the following two conditions holds:

(1) σ and $\#\rho$ are both odd.

(2) σ and $\#\rho$ are both even.

We will also use the following result in the proof of Theorem 2.1.

Lemma 2.4. *Suppose that $\sigma \in S_n$ is a power of an n-cycle. Write $\sigma = (a_1, a_2, \dots, a_n)^m$. Then, the parity of σ is:*

(1) *Even if either n is odd or m is even.*

(2) *Odd if n is even and m is odd.*

Proof. A permutation of n objects can be written as $(n-1)$ transpositions. Taken to the power of m, we now have $m(n-1)$ transpositions. When n is odd or m is even, we have an even number of transpositions, so the permutation will be even. When n is even and m is odd, we have an odd number of transpositions, which means the permutation is odd. □

We now prove our main theorem to determine the parity of the number of components of $T((r_1, s_1), (r_2, s_2))$.

Proof of Theorem 2.1:

The number of components of a twisted torus link $T((r_1, s_1), (r_2, s_2))$ is equal to the component number of its associated permutation.

First, we determine the permutation in S_{r_2} associated to a twisted torus link. Recall that $T((r_1, s_1), (r_2, s_2))$ is the closure of the r_2-strand braid

$$B = (\sigma_1 \sigma_2 \dots \sigma_{r_1 - 1})^{s_1} (\sigma_1 \sigma_2 \dots \sigma_{r_2 - 1})^{s_2},$$

and is constructed by performing s_1 single twists on the first r_1 strands followed by s_2 single twists on all r_2 strands. We determine the permutation associated to $T((r_1, s_1), (r_2, s_2))$ by considering the permutation associated to a single twist. As seen in Figure 4, a single twist sends the first strand to the r^{th} and lowers the index of each other strand. Thus, the permutation associated to a single twist on r strands is $(r, r-1, \dots, 2, 1)$. Therefore, the permutation associated to $T((r_1, s_1), (r_2, s_2))$ is given by $\rho \circ \sigma$, where $\sigma = (r_1, r_1 - 1, \dots, 1)^{s_1}$ and $\rho = (r_2, r_2 - 1, \dots, 1)^{s_2}$. Since we are looking at whether the number of components is odd or even, we are interested in $\#(\rho \circ \sigma) \pmod 2$.

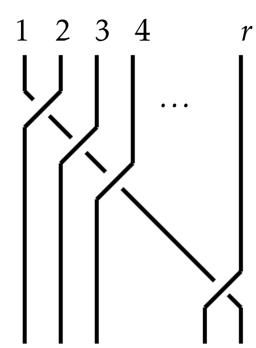

Figure 4. A single twist on r strands.

Consider a twisted torus link $T((r_1, s_1), (r_2, s_2))$. We make several assumptions about the parameters r_1, s_1, r_2, s_2 that do not reduce the scope of the theorem. The sign of s_1 and s_2 in a twisted torus link does not effect the associated permutation or its component number, as σ_i and σ_i^{-1} both map to the same transposition. Therefore we will assume s_1 and s_2 are positive. We also assume that $r_1 < r_2$; if $r_1 = r_2$, then $T((r_1, s_1), (r_2, s_2))$ is the torus link $T(r_1, s_1 + s_2)$. The number of components of $T(p, q)$ can be easily seen to be $gcd(p, q)$. Finally, we assume that $s_1 < r_1$ and $s_2 < r_2$. If $s_i > r_i$, then the permutation associated to $T((r_1, s_1), (r_2, s_2))$ is the same as the permutation associated to $T((r_1, d_1), (r_2, d_2))$, where d_i is the remainder of $s_i \pmod{r_i}$.

We proceed in cases on the parameters of $T((r_1, s_1), (r_2, s_2))$.

Lemma 2.5 (Even Cases)**.** *A twisted torus link $T((r_1, s_1), (r_2, s_2))$ has an even number of components if any of the following hold:*

(1) $r_1 \equiv 1 \pmod 2, r_2 \equiv 0 \pmod 2, s_2 \equiv 0 \pmod 2$

(2) $r_1 \equiv 0 \pmod 2, r_2 \equiv 0 \pmod 2, s_1 \equiv s_2 \pmod 2$

(3) $r_1 \equiv 0 \pmod 2, r_2 \equiv 1 \pmod 2, s_1 \equiv 1 \pmod 2$

Proof. We consider each possible case one at a time. In the first case, since r_1 is odd, by Lemma 2.4, the permutation σ is even regardless of the value of s_1. Since both r_2 and s_2 are even, $\#\rho$, which is equal to the $\gcd(r_2, s_2)$, is even. By Lemma 2.3, the twisted torus link has an even number of components.

In the second case, when s_1 and s_2 are even, by Lemma 2.4, σ is even since s_1 is even, and $\#\rho$ is even because $\gcd(r_2, s_2)$ is even. Once again, by Lemma 2.3 we have an even number of components. On the other hand, we consider the subcase when s_1 and s_2 are odd. Since r_1 is even and s_1 is odd, by Lemma 2.4, σ is odd; since s_2 is odd, $\gcd(r_2, s_2)$ is odd and therefore $\#\rho$ is odd. By Lemma 2.3 there are an even number of components.

In the third case, by Lemma 2.4, σ is odd. Since r_2 is odd, $\gcd(r_2, s_2)$ is odd, which implies that $\#\rho$ is odd. By Lemma 2.3 we once again have an even number of components. □

Thus, whenever a twisted torus link follows one of the cases of Theorem 2.1, it will always have an even number of components. To prove the opposite direction, we will demonstrate that all forms that do not fall under Theorem 2.1 must have an odd number of components.

Lemma 2.6 (Odd Cases)**.** *A twisted torus link $T((r_1, s_1), (r_2, s_2))$ has an odd number of components if any of the following hold:*

(1) $r_1 \equiv 1 \pmod 2$, $s_2 \equiv 1 \pmod 2$

(2) $r_1 \equiv 1 \pmod 2$, $r_2 \equiv 1 \pmod 2$, $s_2 \equiv 0 \pmod 2$

(3) $r_1 \equiv 0 \pmod 2$, $s_1 \equiv 0 \pmod 2$, $s_2 \equiv 1 \pmod 2$

(4) $r_1 \equiv 0 \pmod 2$, $s_1 \equiv 1 \pmod 2$, $r_2 \equiv 0 \pmod 2$, $s_2 \equiv 0 \pmod 2$

(5) $r_1 \equiv 0 \pmod 2$, $s_1 \equiv 0 \pmod 2$, $r_2 \equiv 1 \pmod 2$, $s_2 \equiv 0 \pmod 2$

Proof. In the first case, by Lemma 2.4, σ is even because r_1 is odd. Since s_2 is odd, $\gcd(r_2, s_2)$ is odd, so $\#\rho$ is odd. By Lemma 2.3, the twisted torus link has an odd number of components.

With Case 2, since r_1 is odd, by Lemma 2.4, σ is even. Since r_2 is odd, $\gcd(r_2, s_2)$ is odd and so is $\#\rho$. By Lemma 2.3, we have an odd number of components.

In the third case, by Lemma 2.4, since s_1 is even, σ is even. Since s_2 is odd, $\gcd(r_2, s_2)$ is odd, so $\#\rho$ is odd. By Lemma 2.3, the twisted torus link has an odd number of components.

For Case 4, since r_1 is even and s_1 is odd, σ is odd by Lemma 2.4. Since r_2 and s_2 are even, which causes $\gcd(r_2, s_2)$ to be even, $\#\rho$ is even. By Lemma 2.3, we have an odd number of components.

In the fifth and final case, by Lemma 2.4, since s_1 is even, σ is even. Because $\gcd(r_2, s_2)$ is odd, $\#\rho$ is also odd. By Lemma 2.3, we once again have an odd number of components. □

Therefore, a twisted torus link will have an even number of components if and only if it follows at least one of the conditions stated in Theorem 2.1.

3. Explicit Determination of the Number of Components

While we have not yet been able to explicitly determine the number of components of a twisted torus link, we have made some specific progress in this direction.

Theorem 3.1. *Consider a twisted torus link of the form* $T((r_1, s), (r_2, \pm s))$ *with* $s \equiv 0 \pmod 2$. *Then,*

(1) *If* $r_2 \equiv \pm 1 \pmod s$ *and* $r_1 \equiv s \pmod{2s}$, *the twisted torus link will have* 1 *component and is therefore a knot.*

(2) *If* $r_2 \equiv \pm 1 \pmod s$ *and* $r_1 \equiv 0 \pmod{2s}$, *the twisted torus link will have* $s + 1$ *components.*

Because we are looking at even values of s, and $s + 1$ and $s - 1$ influence the number of components the twisted torus link will have, we run into an interesting corollary that involves pairs of twin primes.

Corollary 3.2 (Twin Primes)**.** *Given a twisted torus link* $T((r_1, s), (r_2, \pm s))$ *with* $s - 1$ *and* $s + 1$ *prime and* $r_2 \equiv \pm 1 \pmod s$, *the twisted torus link will have one component if* $r_1 \not\equiv s \mp 1 \pmod{2s}$. *Otherwise, it will have* $s - 1$ *components when* $r_1 \equiv s \mp 1 \pmod{2s}$

Proof of Theorem 3.1:

Proof. To prove part 1 of Theorem 3.1, we note that the assumption that $r_1 \equiv s \pmod{2s}$ guarantees that the first part of the twisted torus link will consist of some number of full twists on the first r_1 strands. Therefore, the component number of the twisted torus link will only depend on the permutation of the s twists on all r_2 strands. This number is

$\gcd(r_2, s)$. Since $r_2 \equiv \pm 1 \pmod{s}$, there exists some integer k such that $r_2 = ks \pm 1$, and we see that $\gcd(r_2, s) = \gcd(ks \pm 1, s) = 1$. Thus, the twisted torus link will be a knot.

To prove part 2, we rely on a result from [4], showing that under certain conditions on r_1, s_1, r_2, s_2, the twisted torus link $T((r_1, s_1), (r_2, s_2))$ decomposes as a split link comprised of two disjoint torus links.

Proposition 3.3 (Quoted from [4]). *Consider the twisted torus link $T((r_1, s_1), (r_2, -s_2))$ where $r_1, s_1, r_2, s_2 \in \mathbb{N}$ with $r_2 \geq r_1 \geq s_1 + s_2$ and $r_1 \bmod (s_1 + s_2) = 0$. Then $T((r_1, s_1), (r_2, -s_2))$ is isotopic to $T(s_2, r_1 - r_2 - s_2 \frac{r_1}{s_1+s_2}) \sqcup T(s_1, s_1 \frac{r_1}{s_1+s_2})$.*

Thus, the component number of the twisted torus link $T((r_1, s), (r_2, \pm s)$ will be the sum of the component numbers of the torus links $T(s, r_1 - r_2 - s\frac{r_1}{2s})$ and $T(s, s\frac{r_1}{2s})$.

The torus link $T(s, r_1 - r_2 - s\frac{r_1}{2s})$ has $\gcd(s, r_1 - r_2 - s\frac{r_1}{2s})$ components and the torus link $T(s, s\frac{r_1}{2s})$ has $\gcd(s, s\frac{r_1}{2s})$ components. From our assumptions on r_1 and r_2, we know that there exist integers k_1, k_2 such that $r_1 = 2k_1 s$ and $r_2 = k_2 s \pm 1$. Therefore, $r_1 - r_2 - s\frac{r_1}{2x} = 2k_1 s - (k_2 s \pm 1) - sk_1 = k_1 s - k_2 s \mp 1$. Thus, $\gcd(s, r_1 - r_2 - s\frac{r_1}{2s}) = \gcd(s, (k_1 - k_2)s \mp 1) = 1$, and $\gcd(s, s\frac{r_1}{2s}) = \gcd(s, sk_1) = s$. Therefore, the twisted torus link $T((r_1, s), (r_2, \pm s)$ has $s + 1$ components. □

4. Conjectures and Further Discussion

We made some progress in generalizing Theorem 3.1, but have not yet developed a formal proof. However, we wrote a computer program that demonstrated that they hold for tens of thousands of examples.

Conjecture 4.1. *Given a twisted torus link of the form $T((r_1, s), (r_2, \pm s))$ with $s \equiv 0 \pmod{2}$:*

1. *If $r_2 \equiv 1 \pmod{s}$ and $1 \leq r_1 \pmod{2s} \leq s$, then the twisted torus link will have $n = \gcd(r_1, s-1)$ components.*
2. *If $r_2 \equiv 1 \pmod{s}$ and $s+1 \leq r_1 \pmod{2s} \leq 2s-1$, then the twisted torus link will have $n = \gcd(r_1 + 2, s+1)$ components.*
3. *If $r_2 \equiv -1 \pmod{s}$ and $1 \leq r_1 \pmod{2s} \leq s$, then the twisted torus link will have $n = \gcd(r_1, s+1)$ components.*
4. *If $r_2 \equiv -1 \pmod{s}$ and $s+1 \leq r_1 \pmod{2s} \leq 2s-1$, then the twisted torus link will have $n = \gcd(r_1 - 2, s-1)$ components.*

Conjecture 4.2. *Given a twisted torus link of the form $T((r_1, s), (r_2, \pm s))$ with $s \equiv 0 \pmod{2}$, the following statements are true:*

1. *If $r_1 \equiv 1, s, 2s-1 \pmod{2s}$ and $r_2 \equiv 0 \pmod{s}$, then the twisted torus link will have s components.*

Conjecture 4.3. *Given a twisted torus link of the form $T((r_1, s), (r_2, \pm s))$ with $s \equiv 1 \pmod{2}$, the following statements are true:*

1. *If $r_1 \equiv 0 \pmod{2s}$ and $r_2 \not\equiv 0 \pmod{s}$, then the twisted torus link will have $s+1$ components.*

(2) *If* $r_1 \equiv 2, 2s-2 \pmod{2s}$ *and* $r_2 \equiv 0 \pmod{s}$, *then the twisted torus link will have* $s-1$ *components.*

(3) *If* s *is a prime,* $r_1 \equiv 1, s, 2s-1 \pmod{2s}$ *and* $r_2 \not\equiv 0 \pmod{s}$, *then the twisted torus link will have* 1 *component.*

The approach we take in proving Theorem 2.1 is algebraic, using the permutation associated to a twisted torus link. This algebraic approach is useful in determining component number, as the positioning of the strands in a braid determine its component number. However, a further direction that may be promising is to use a geometric approach. In order to work toward a full classification of twisted torus links, we will need insight not only to the number of components, but also to how the components interact within the link. There are many linking invariants that could prove useful in this direction. In considering the geometry of a twisted torus link, we made some progress in tracking specific strands of the braid throughout the diagram. While there is some tricky number theory at play, adjacent strands in the braid travel in "chunks," with the size of the chunks depending on the parameters of the twisted torus link. We looked to define a mapping formula for the j^{th} strand of $T((r_1, s_1), (r_2, s_2))$ that will depend on the value of j in relation to s_1, r_1, and r_2.

Using this approach, we were able to see some characteristics that hold true for any $T((r_1, s_1), (r_2, s_2))$. Figure 5 shows the exact formulas for the mapping of any strand j in the braid representation of a twisted torus link.

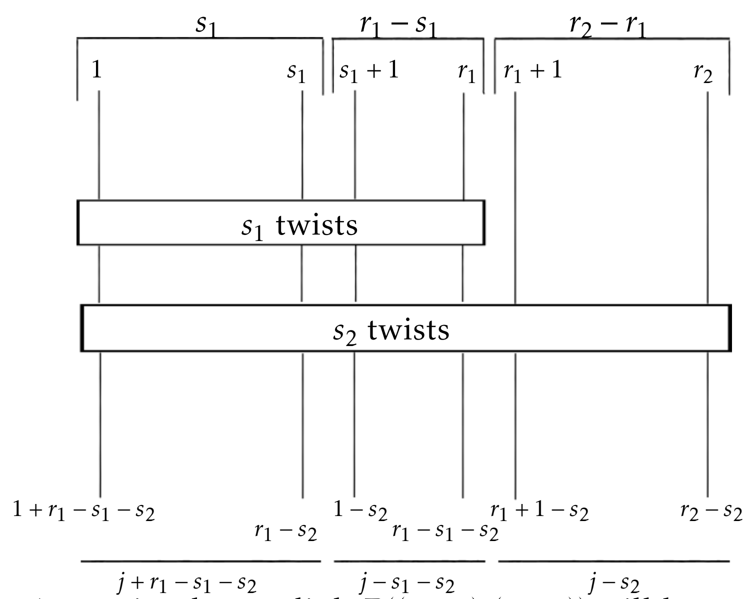

FIGURE 5. Any twisted torus link $T((r_1, s_1), (r_2, s_2))$ will have this mapping. There are three different formulas for mapping strand j, depending on where j lies in relation to s_1, r_1, and r_2. All formulas represent equivalence classes $\pmod{r_2}$.

From this figure, we see where any given strand j in the braid representation maps depending on where it lies in relation to s_1, r_1, and r_2. Note these formulas show that any given "chunk" of strands with the same mapping formula will map to sequential equivalence classes $\pmod{r_2}$ and that all the strands in each chunk will go to each class in order, starting with the first chunk then the third chunk and ending with the last chunk. A couple special scenarios follow immediately from Figure 5. When $s_1 + s_2 = r_1$, the twisted torus link will have $s_1 + \gcd(r_2 - s_1, s_2)$ components, since the first s_1 strands will map back to themselves and the rest will follow the pattern of a torus link. Further, when $s_1 + s_2 = r_2$, the twisted torus link will have $r_1 - s_1 + \gcd(r_2 - r_1 + s_1, s_1 \pmod{r_2 - r_1 + s_1})$ components, since the strands between s_1 and $r_1 + 1$ will map back to themselves, and the rest of the strands will behave as a torus link.

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References

[1] Colin Adams. *The knot book: an elementary introduction to the mathematical theory of knots*. American Mathematical Society, Providence, Rhode Island, 2004.

[2] Joan Birman and Ilya Kofman. A new twist on Lorenz links. *Journal of Topology*, 2(2):227–248, 2009.

[3] John Dean. Small Seifert-fibered Dehn surgery on hyperbolic knots. *Alg Geom Top*, 3:435–472, 2003.

[4] Eric Hanson and Max Zhou. Twisted torus links and the general braid. *In Preparation*.

[5] Sangyop Lee. Twisted torus knots that are unknotted. *International Mathematics Research Notices*, 2014(18):4958–4996, 2014.

[6] Kanji Morimoto. On composite twisted torus knots. *Tokyo J. of Math.*, 35(2):499–503, 12 2012.

Student biographies

Michelle S. Berry: Michelle S. Berry will graduate with a Bachelor of Arts in Mathematics and Physics from Goucher College in May 2017. She will start her PhD in Physics at the University of Massachusetts Amherst in the fall of 2017 and plans on focusing on High Energy Theory.

Victoria Diaz: Victoria graduated with a Bachelor's in mathematical sciences from Florida International University in 2016 and will complete her Master's in mathematical sciences in May 2017. She will begin her doctoral studies in industrial engineering at the University of Washington in the fall of 2017.

Emily T. Winn: (*Corresponding author:* etwinn17@g.holycross.edu) Emily will graduate with a Bachelor of Arts in Mathematics from the College of the Holy Cross in May 2017. She will enter a PhD program in Applied Mathematics at Brown University in the fall of 2017.

Max Zhou: Max graduated from Indiana University Bloomington in May 2016 with a Bachelors of Science in mathematics and a minor in computer science. He is currently a mathematics PhD student at the University of California, Los Angeles.